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TECHNICAL NOTE #5

Computation of the Emissivity
of a Cylindrically Symmetric Light Source
from Measurements of the Projected
Intensity Profile

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COMPUTATION OF THE EMISSIVITY OF A CYLINDRICALLY SYMMETRIC LIGHT SOURCE FROM MEASUREMENTS OF THE PROJECTED INTENSITY PROFILE

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A method is described for computing the emissivity ~~$f(r)$ ($r\leq 1$)~~ of a cylindrically symmetric, optically thin light source, when the projected intensity profile ~~$I(x) = \int f(r)\,dy$~~ is known from experiment. The unknown function ~~$f(r)$~~ is taken as a series expansion in terms of orthogonal polynomials, and it is shown how the expansion coefficients can be determined, ~~from $I(x)$.~~ This procedure yields a least-squares smoothed approximation, ~~for $f(r)$~~.

Let $f(r)$ be the emission coefficient (emission per unit area) of a cylindrically symmetric, optically thin light source of unit radius. Side-on measurements of the emission give the projected intensity profile $I(x)$, which is related to $f(r)$ by

$$I(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(r)\,dy = 2\int_x^1 \frac{f(r)\,r\,dr}{(r^2-x^2)^{\frac{1}{2}}} \qquad (1)$$

The inversion of eq. (1), giving $f(r)$ in terms of $I(x)$, is

$$f(r) = -\frac{1}{\pi}\int_r^1 \frac{I'(x)\,dx}{(x^2-r^2)^{\frac{1}{2}}} = -\frac{2}{\pi}\,\frac{d}{d(r^2)}\int_r^1 \frac{I(x)\,x\,dx}{(x^2-r^2)^{\frac{1}{2}}}. \qquad (2)$$

The problem of determining $f(r)$ when $I(x)$ is known from experiment has been treated by several authors,[1-3] mostly by some method of strip integration. An accurate numerical method has been given by Bockasten.[2]

The x-axis is divided in a number of equal intervals, and a third-degree polynomial is fitted to the $I(x)$ curve within each interval. Eq. (2) is used, and a matrix A_{ik} is obtained that transforms a set of values of I to values of f,

$$f(r_i) = \sum_k A_{ik} I(x_k). \tag{3}$$

Bockasten also discusses the influence of random errors in the values of $I(x)$.

The method described in this note should be advantageous whenever the $I(x)$ curve has a more or less irregular shape, and a properly smoothed approximation to $f(r)$ is desired. The unknown function $f(r)$ is expanded in a series of orthogonal polynomials. The expansion coefficients can be determined from $I(x)$, and in this way a least-squares smoothed approximation to $f(r)$ can be found directly.

A few general consequences of eqs. (1) and (2) are of interest. Obviously $I(x)$ is an even function, and we consider the interval $0 \leq x \leq 1$ only. The behavior of $f(r)$ near $r = 1$ is very sensitive to the behavior of $I(x)$ near $x = 1$. If $I(x)$ is proportional to $(1-x)^b$ near $x = 1$, then $f(r)$ will be proportional to $(1-r)^{b-\frac{1}{2}}$ near $r = 1$. For instance, $b = 0.49$, $b = 0.5$, and $b = 0.51$ correspond to infinite, finite, and zero values of $f(1)$, respectively. Further, it is seen from eq. (2) that a discontinuity in $I'(x)$ at $x = 0$ implies an infinite value of $f(0)$. We will assume that the functions $I(x)$ and $f(r)$ are finite, so that $I'(0) = 0$, and $I(1) = 0$ (with $b \geq \frac{1}{2}$).

We now choose to fit an even polynomial $f_N(r)$, of degree [illegible], to $f(r)$ in such a way that the error squared, integrated over the cross-section $r \leq 1$, is minimized; thus,

$$\int_0^1 (f - f_M)^2 \, r \, dr = \text{min.}, \ M \text{ fixed.} \tag{4}$$

The orthogonal polynomials suitable for this purpose are the Legendre polynomials [4] $P_m(t)$, with $t = 2r^2 - 1$. The first few of these are $P_o = 1$, $P_1 = 2r^2 - 1$, $P_2 = 6r^4 - 6r^2 + 1$, $P_3 = 20r^6 - 30r^4 + 12r^2 - 1$. They satisfy the orthogonality relation

$$2(2m+1) \int_0^1 P_n(2r^2 - 1) \, P_m(2r^2 - 1) \, r \, dr = \delta_{mn}, \tag{5}$$

so that the series expansion of $f(r)$ is

$$f(r) = \sum_{m=o}^{\infty} a_m \, P_m(2r^2 - 1) \tag{6}$$

with

$$a_m = 2(2m+1) \int_0^1 f(r) \, P_m(2r^2 - 1) \, r \, dr. \tag{7}$$

If the series (6) is terminated at $m = M$, the result is a 2M-degree polynomial f_M, which satisfies the condition (4).

We substitute the series (6) in eq. (1), integrate term by term, and obtain[x)]

$$I(x) = \sum_{m=o}^{\infty} 2(2m+1)^{-1} a_m \sin\left[(2m+1)\cos^{-1} x\right], \tag{8}$$

or, putting $x = \cos\theta$,

$$I(\cos\theta) = \sum_{m=o}^{\infty} 2(2m+1)^{-1} a_m \sin(2m+1)\theta. \tag{9}$$

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x) The functions $U_n(x) \equiv \sin(n \cos^{-1} x)$ are related to the Chebyshev polynomials $T_n(x) \equiv \cos(n \cos^{-1} x)$. They satisfy the orthogonality relation $\int_{-1}^{1} U_k U_n (1-x^2)^{-\frac{1}{2}} dx = 0$, $k \neq n$, and vanish at the points $x = \pm 1$. $U_n(x)$ is an even function if n is odd.[4]

It can be shown from eqs. (1) and (7) that

$$2(2m+1)^{-1}\, a_m = (4/\pi) \int_0^{\pi/2} I(\cos\theta)\, \sin(2m+1)\theta \;\; d\theta, \tag{10}$$

so that eq. (1) (with $x = \cos\theta$) transforms the Legendre series of $f(r)$ into the Fourier series of $I(\cos\theta)$ (odd sine terms only). It may be noted that the total emission from the light source is

$$2\pi \int_0^1 f(r)\, r\, dr = 2 \int_0^1 I(x)\, dx = \pi a_o. \tag{11}$$

It is also seen that, if $f_M(r)$ satisfies the condition (4), then the corresponding function $I_M(x)$ (the series (8) terminated at $m = M$) will be the "best" approximation for $I(x)$, in the sense that the integral

$$\int_0^1 (I - I_M)^2\, (1 - x^2)^{-\frac{1}{2}}\, dx$$

is minimized.

The standard methods of Fourier analysis [5] can now be used for determining the coefficients a_m. One possible procedure is as follows: The series (9) is terminated at some $m = N$:

$$I_N(\cos\theta) = \sum_{m=0}^{N} 2(2m+1)^{-1}\, a_m \sin(2m+1)\theta, \tag{12}$$

and I_N is required to coincide with I in those points in the interval $0<\theta<\pi/2$ where the first neglected term ($\sin(2N+3)\theta$) vanishes. These points are

$$\theta_k = k\pi/(2N+3), \quad k = 1, 2, \ldots, N+1.$$

and the result is

$$a_m = 2\,\frac{2m+1}{2N+3} \sum_{k=1}^{N+1} I(\cos\frac{k\pi}{2N+3})\,\sin\frac{(2m+1)k\pi}{2N+3}, \quad m=0,\ 1\ldots N. \qquad (13)$$

This result can also be obtained by evaluating the integral (10) by the trapezoidal rule. Eq. (13) is exact if $f(r)$ is an even polynomial of degree $2(N+1)$ or less, so that the series (6), (8), and (9) contain no terms beyond $m = N+1$.

With the a_m given by eq. (13), the 2M-degree polynomial approximation for $f(r)$ becomes

$$f_M(r) = \sum_{m=o}^{M} a_m\,P_m(2r^2-1) = \sum_{k=1}^{N+1} A_k(r)\,I(\cos\frac{k\pi}{2N+3}) \qquad (14)$$

where

$$A_k(r) = \frac{2}{2N+3} \sum_{m=o}^{M} (2m+1)\,\sin\frac{(2m+1)k\pi}{2N+3}\,P_m(2r^2-1). \quad (M\leq N) \qquad (15)$$

As an example we consider the function

$$I = (1 - x^2)^2 = \sin^4\theta, \qquad (16)$$

corresponding to

$$f(r) = (8/3\pi)(1-r^2)^{3/2}. \qquad (17)$$

The coefficients a_m, computed from eq. (13), are listed in Table 1. The exact a_m, given by eq. (7) or (10) are entered in the last line. They are

$$a_m = \frac{48}{\pi}\,\frac{1}{(2m-3)(2m-1)(2m+3)(2m+5)}, \qquad (18)$$

showing fairly rapid convergence. The convergence is slowest near the points $r=0$ and $r=1$, where $|P_m| = 1$. A seven-point analysis ($N=6$) is sufficient to reduce the error in $f(r)$ to about 0.0006 for $r=0$ and 0.001 for $r=1$.

Table 2 shows the same for

$$I = 1-x^2 = \sin^2\theta, \tag{19}$$

corresponding to

$$f(r) = (2/\pi)\,(1-r^2)^{\frac{1}{2}}. \tag{20}$$

The exact coefficients a_m are

$$a_m = -\frac{4}{\pi}\,\frac{1}{(2m-1)(2m+3)}, \tag{21}$$

so that the convergence is fairly slow. Clearly, the behavior of the function (20) near $r=1$ cannot be represented accurately by a low-degree polynomial. However, a seven-point analysis ($N=6$) suffices to give an error less than about 0.002 for $r \leq 0.98$.

Experimental measurements will commonly yield $I(x)$ curves with irregular fluctuations, rather than smooth functions of the type (16) or (19). In such cases it should be an advantage of this method that, with a suitable choice of N and M, a properly smoothed $f(r)$ is obtained directly.

The influence of random errors in the values of $I(x)$ can be found as follows. If the error (standard deviation) of I at each point is ΔI, eq. (13) gives for the error Δa_m of a_m,

$$(\Delta a_m)^2 = 4\left(\frac{2m+1}{2N+3}\right)^2 (\Delta I)^2 \sum_{k=1}^{N+1} \sin^2 \frac{(2m+1)k\pi}{2N+3} = \frac{(2m+1)^2}{2N+3}\,(\Delta I)^2,$$

or

$$\Delta a_m = \frac{2m+1}{(2N+3)^{\frac{1}{2}}}\,\Delta I. \tag{22}$$

For the function (19), assuming $\Delta I = 0.002$ and $N = 6$, we see from eq. (22) and Table 2 that the error in a_5 is comparable to a_5 itself, so that this source of error is more important than the fairly slow convergence of the Legendre series.

From eq. (14), the error in f_i is given by

$$(\Delta f_M)^2 = (\Delta I)^2 \sum_{k=1}^{N+1} \left[A_k(r)\right]^2, \tag{23}$$

which cannot be expressed in a simple form. We can obtain an estimate for Δf_M from the expression

$$(\Delta f_M)^2 = \sum_{m=0}^{M} (\Delta a_m)^2 \, P_m^{\,2}, \tag{24}$$

which would be correct if the Δa_m were uncorrelated. Δf_M will be largest at the points $r=0$ and $r=1$, where $|P_m| = 1$. We find from eqs. (22) and (24),

$$\Delta f_M = \Delta I \sqrt{\frac{(M+1)(2M+1)(2M+3)}{3(2N+3)}}, \quad r=0,\ 1. \tag{25}$$

For intermediate values of r, the error will be smaller, although no simple formula can be given. For instance, for $r^2 = 0.5$ and $M = 6$, eqs. (22) and (24) yield $\Delta f_6 = 5.9 \times (2N+3)^{-\frac{1}{2}} \Delta I$, while the end-point value given by eq. (25) is $21 \times (2N+3)^{-\frac{1}{2}} \Delta I$. For $M = 10$ the factor is 9.0 and 42, respectively.

Acknowledgement

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4. See, for example, E. Madelung, Die Mathematischen Hilfsmittel des Physikers, 5th ed. (Springer, 1953)
5. See, for example, E. Whittaker and G. Robinson, The Calculus of Observations, 4th ed. (Blackie & Son, 1944)

Table 1

Expansion coefficients a_m of the function $(8/3\pi)(1 - r^2)^{3/2}$, approximate values from eq. (13) with $N = 0, 1, 3, 6$, and (last line) correct values from eq. (18).

N	a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	0.32476						
1	0.33931	−0.44089					
3	0.33957	−0.43660	0.08062	0.00951			
6	0.33953	−0.43657	0.08083	0.01027	0.00306	0.00117	0.00046
(∞)	0.33953	−0.43654	0.08084	0.01029	0.00305	0.00124	0.00061

Table 2

Expansion coefficients a_m of the function $(2/\pi)(1 - r^2)^{\frac{1}{2}}$; approximate values from eq. (13) with $N = 0, 1, 3, 6$, and (last line) correct values from eq. (21).

N	a_0	a_1	a_2	a_3	a_4	a_5	a_6
0	0.4330						
1	0.4253	−0.2437					
3	0.4245	−0.2538	−0.0580	−0.0216			
6	0.4244	−0.2546	−0.0603	−0.0277	−0.0154	−0.0089	−0.0044
(∞)	0.4244	−0.2546	−0.0606	−0.0283	−0.0165	−0.0109	−0.0077

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